FRACTIONAL DERIVATIVE OF THE POLYNOMIALS IN THE SPACE OF FOUR-DIMENSIONAL NUMBERS

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Abstract
We explore the use of power series in solving generic polynomial differential equations in four-dimensional space, employing the Caputo fractional derivative. As is widely recognized, power series transform a continuous formulation into a discrete system of difference equations, which can be efficiently solved through recursive methods. The key contribution of this work lies in rigorously proving the convergence of these series in a neighborhood of the initial point. The theory of four-dimensional functions is the new method in mathematics and due to this is scantily explored.

Keywords: Four-dimensional space, power series, convergence, radius of convergence

ДРОБНАЯ ПРОИЗВОДНАЯ МНОГОЧЛЕНОВ В ПРОСТРАНСТВЕ ЧЕТЫРЕХМЕРНЫХ ЧИСЕЛ

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Аннотация
Мы исследуем использование степенных рядов для решения общих полиномиальных дифференциальных уравнений в четырехмерном пространстве, используя дробную производную Капуто. Как широко известно, степенные ряды преобразуют непрерывную формулировку в дискретную систему разностных уравнений, которая может быть эффективно решена с помощью рекурсивных методов. Ключевой вклад этой работы заключается в строгом доказательстве сходимости этих рядов в окрестности начальной точки. Теория четырехмерных функций является новым методом в математике и поэтому мало изучена.

Ключевые слова: четырехмерное пространство, степенные ряды, сходимость, радиус сходимости

Introduction

The study of power series has long been a cornerstone of mathematical analysis, enabling us to represent a wide range of functions as infinite sums of terms. In 2015 it was possible to describe the initial chapters of this theory by Kazakh researcher M.M.Abenov [1-2]. The discovery of the spectral theory made it possible to systematize the available results and obtain new information about theory of four-dimensional variables and solved three dimensional problems analytically [2-3]. Full substantiated analysis of this theory was published by Abenov M. (Al-Farabi Kazakh National University). While developing this theory, Abenov M.M. and Gabbasov M.B. found all four-dimensional Abenov spaces with commutative multiplication, which were assigned the designations M2-M7, and it became necessary to study these spaces. In this paper we performed a research in Abenov space of four-dimensional numbers M3 [4-5].

The theory of four-dimensional functions is based on the principles of commonality of its key concepts with similar categories of one-dimensional and two-dimensional analysis [2]. This approach is justified by the fact that linear spaces of one-dimensional and two-dimensional numbers can initially be considered as eigenspaces of a more general space of four-dimensional numbers.

This leads to the study of these one-dimensional, two-dimensional and four-dimensional numbers as elements of a triune system, which allows to uniformly define key concepts in the theory of these numbers (for example, arithmetic operations on the set of these numbers) [2-5]. Note that a similar approach was previously used by the physicist W. Hamilton when he constructed the quaternion algebra. He defined a non-commutative operation of multiplication of elements, which allowed avoiding zero divisors [2].

Traditionally, power series have been extensively studied in the context of real and complex numbers. However, with the advent of higher-dimensional mathematics, the extension of power series to four-dimensional numbers presents intriguing possibilities.

Let consider a linear space

$$\mathcal{R}^4 = \left\{ X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} : x_k \in \mathbb{R}, k = \overline{1,4} \right\}$$

where the addition and multiplication by scalar is defined like in usual Euclidean space R^4.

Let consider a polynomial

$$p(X) = \sum_{\alpha_1=0}^{n_1} \sum_{\alpha_2=0}^{n_2} \sum_{\alpha_3=0}^{n_3} \sum_{\alpha_4=0}^{n_4} b_{\alpha_1\alpha_2\alpha_3\alpha_4} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} x_4^{\alpha_4}$$

By using the a multi index $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, $X^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} x_4^{\alpha_4}$ we can write it as follows

$$p(X) = \sum_{|\alpha|=0}^{n} b_\alpha X^\alpha$$

where we use notation $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ and $n = n_1 + n_2 + n_3 + n_4$.

For example, when $n = 2$ we have

$$\sum_{|\alpha|=0}^{2} b_\alpha X^\alpha = b_{0000} + b_{1000}x_1 + b_{0100}x_2 + b_{0010}x_3 + b_{0001}x_4 + b_{2000}x_1^2 + b_{1100}x_1x_2 + b_{1010}x_1x_3 + b_{1001}x_1x_4 + b_{0200}x_2^2 + b_{0110}x_2x_3 + b_{0101}x_2x_4 + b_{0020}x_3^2 + b_{0011}x_3x_4 + b_{0002}x_4^2$$

Let X, $Y \in$ R^4. We define the product of these numbers as follows:

$$X \cdot Y = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} x_1y_1 - x_2y_2 + x_3y_3 - x_4y_4 \\ x_2y_1 + x_1y_2 + x_4y_3 + x_3y_4 \\ x_3y_1 - x_4y_2 + x_1y_3 - x_2y_4 \\ x_4y_1 + x_3y_2 + x_2y_3 + x_1y_4 \end{pmatrix}$$

Such defined the product of two four dimensional numbers satisfy the properties of commutativity and associativity over addition. The module of the four dimensional number is defined by the following formula

$$|X| = \sqrt{[(x_1 - x_3)^2 + (x_2 - x_4)^2] \cdot [(x_1 + x_3)^2 + (x_2 + x_4)^2]}$$

Power series of four-dimensional numbers

We denote by Λ_X a spectrum of the X $\in$ R^4:

$$\Lambda_X = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix},$$

where

$$\begin{cases} \lambda_1 = x_1 - x_3 + (x_2 - x_4) \cdot i \\ \lambda_2 = x_1 - x_3 - (x_2 - x_4) \cdot i \\ \lambda_3 = x_1 + x_3 + (x_2 + x_4) \cdot i \\ \lambda_2 = x_1 + x_3 - (x_2 + x_4) \cdot i \end{cases}$$

We define the spectral norm of the X $\in$ R^4 as

$$\|X\|_\Lambda = \frac{1}{4}\sum_{k=1}^{4} |\lambda_k|$$

The open ball centered at $X_0 \in$ R^4 with the radius $r > 0$, is defined by the following:

$$B(X_0, r) = \{X \in \mathcal{R}^4 : \|X - X_0\|_\Lambda < r\}.$$

Let define a power series in R^4:

$$p(X) = \sum_{k=0}^{\infty} c_k \cdot (X^\alpha)^k = \sum_{k=0}^{\infty} c_k \cdot X^{\alpha k} \qquad (2)$$

Where c_k are four dimensional constants, and X is a four-dimensional variable. Now we define the powers of the polynomials:

$$p^2(X) = \sum_{m=0}^{\infty}\left(\sum_{k=0}^{m} c_k c_{m-k}\right)\cdot X^{\alpha m}$$

$$p^3(X) = \sum_{n=0}^{\infty}\left[\sum_{m=0}^{n}\left(\sum_{k=0}^{m} c_k c_{m-k}\right)c_{n-m}\right]\cdot X^{\alpha n},$$

In general case we have

$$p^{\lambda}(X) = \sum_{n_\lambda=0}^{\infty}\left[\sum_{n_{\lambda-1}=0}^{n_\lambda}\cdots\sum_{n_2=0}^{n_3}\left(\sum_{n_1=0}^{n_2} c_{n_1}c_{n_2-n_1}\right)c_{n_3-n_2\ldots.}c_{n_\lambda-n_{\lambda-1}}\right]\cdot X^{\alpha n_N}$$

By using the notation

$$\Theta_\lambda(c_0, c_1, \ldots, c_n) = \sum_{n_{\lambda-1}=0}^{n_\lambda}\cdots\sum_{n_2=0}^{n_3}\left(\sum_{n_1=0}^{n_2} c_{n_1}c_{n_2-n_1}\right)c_{n_3-n_2\ldots.}c_{n_\lambda-n_{\lambda-1}}$$

we have

$$p^{\lambda}(X) = \sum_{n=0}^{\infty}\Theta_\lambda(c_0, c_1, \ldots, c_n)\cdot X^{\alpha n}$$

We define fractional differentiation of powers

$$\mathcal{D}^{\alpha}X^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}X^{\gamma-\alpha}, \gamma > -1.$$

The power series (2) is differentiated, term by terms as follows

$$\mathcal{D}^{\alpha}p(X) = \sum_{k=0}^{\infty} c_k\cdot\mathcal{D}^{\alpha}(X^{\alpha k}) = \sum_{k=0}^{\infty} c_{k+1}\cdot\frac{\Gamma\big((k+1)\alpha+1\big)}{\Gamma(k\alpha+1)}X^{\alpha k} \qquad (3)$$

When the power series (2) is convergent in the interval [0, *l*), then the power series for fractional derivative (3) also converges for [0, *l*).

Theorem

There exists a non-negative real number $\epsilon > 0$, such that the power series (2) and (3) converges for all $X \in B(0,\epsilon)$.

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